Neutrino Oscillations in Caianiello's Quantum Geometry Model

V. Bozza[a,b]*, S Capozziello[a,b], G. Lambiase[a,b], and G. Scarpetta[b,c,d]
[a] *Dipartimento di Scienze Fisiche "E.R. Caianiello", Università di Salerno, 84081 Baronissi (Sa), Italy.*
[b] *Istituto Nazionale di Fisica Nucleare, Sez. di Napoli, Italy.*
[c] *Dipartimento di Fisica, Universitá di Salerno, 84081 Baronissi (Sa), Italy*
[d] *IIASS, Vietri sul Mare (Sa), Italy.*

June 22, 2001

Abstract

Neutrino flavor oscillations are analyzed in the framework of Quantum Geometry model proposed by Caianiello. In particular, we analyze the consequences of the model for accelerated neutrino particles which experience an effective Schwarzschild geometry modified by the existence of an upper limit on the acceleration, which implies a violation of the equivalence principle. We find a shift of quantum mechanical phase of neutrino oscillations, which depends on the energy of neutrinos as E^3. Implications on atmospheric and solar neutrinos are discussed.

PACS: 04.70.-s, 04.70.Bw
Keywords: Neutrino Oscillations, Quantum Geometry, Maximal Acceleration

*E-mail: valboz,capozziello,lambiase,scarpetta@sa.infn.it

1 Introduction

The long–standing problem about the deficiency of the solar neutrino and the atmospheric neutrino problem might be explained invoking oscillations between the various flavors or generations of neutrinos. In fact, neutrino oscillations can occur in the vacuum if the eigenvalues of the mass matrix are not all degenerate, and the corresponding mass eigenstates are different from weak interaction eigenstates (Bilenky and Pontecorvo, 1978). The most discussed version of this type of solutions is the MSW effect (Mikhyev and Smirnov, 1986a, 1986b; Wolfeinstein, 1978) in which solar electron neutrinos are converted almost completely in muon or tau neutrinos owing to the presence of matter in the Sun.

Recently, the quantum mechanical oscillations of neutrinos propagating in a gravitational field (usually the Schwarzschild field) have been discussed by several authors (see for example (Ahluwalia and Burgad, 1996; Bhattacharya *et al.*, 1999) and reference therein), also in view of astrophysical consequences. Ahluwalia and Burgard consider in fact, the gravitational effect on the oscillations showing that an external weak gravitational field of a star adds a new contribution to the phase difference (Ahluwalia and Burgard, 1996). They also suggest that the new oscillation phase may be a significant effect on the supernova explosions since the extremely large fluxes of neutrinos are produced with different energies corresponding to the flavor states. This result has been also discussed by Bhattachya, Habib and Mottola (Bhattacharya *et al.*, 1999). In their approach it is shown that the possible gravitational effect appears at the higher order with respect to one calculated in Ref. (Ahluwalia and Burgard, 1996), with a magnitude of the order 10^{-9}, which is negligible in typical astrophysical applications.

An alternative mechanism of neutrino oscillations has been proposed in (Gasperini, 1998, 1989; Halprin and Leung, 1991) as a means to test the equivalence principle. In this mechanism, neutrino oscillations follow by assuming a flavor non–diagonal coupling of neutrinos to gravity which violates the equivalence principle, i.e. if the universality of the gravitational couplings to different flavors breaks down, additional phase difference appears. Therefore, the understanding how the presence of a gravitational field or the violation of the equivalence principle affects the neutrino oscillation phase is an important matter.

In this paper we face this issue in the framework of Quantum Geometry model proposed by Caiainiello some years ago in the attempt to unify Quantum Mechanics and General Relativity principles (see (Caianiello, 1981, 1992) and references therein). In this model the effective four–dimensional metric depends on the mass of a given test particle, so that *test particles with different rest masses experience different geometries and, as consequence, an effective violation of the equivalence principle occurs.* The geodesic paths along which the test particles are moving become mass–dependent, resulting in a non–universality of the gravitational coupling (Caianiello *et al*, 1990), and making the metric observer–dependent, as also conjectured by Gibbons and Hawking (Gibbons and Hawking, 1977).

The view frequently held that the proper acceleration of a particle is limited upwardly (Caianiello *et al.*, 1982) finds in this model a geometrical interpretation epitomized by the line element

$$d\tilde{s}^2 = \left(1 + \frac{g_{\mu\nu} d\ddot{x}^\mu d\ddot{x}^\nu}{\mathcal{A}_m^2}\right) ds^2 \equiv \sigma^2(x) ds^2 \,, \tag{1}$$

experienced by the accelerating particle along its worldline. In (1) $\mathcal{A}_m = 2mc^3/\hbar$ is the proper Maximal Acceleration (MA) of the particle of mass m and $\ddot{x}^\mu$ its four–acceleration.

MA has several implications. It provides a regularization method in Quantum Field Theory (Feoli *et al.*, 1999b), allowing to circumvent inconsistencies associated with the application of the point-like concept to relativistic quantum particles, it is the same cut–off on the acceleration required in an *ad hoc* fashion by Sanchez in order to regularize the entropy and the free energy of quantum strings (Sanchez, 1993), and it is also invoked as a necessary cut–off by McGuigan in the calculation of black hole entropy (McGuigan 1994).

Applications of Caianiello's model include cosmology (Caianiello *et al.*, 1991; Capozziello *et al.*, 1999), where the initial singularity can be avoided while preserving inflation, the dynamics of accelerated strings (Feoli, 1993) and the energy spectrum of a uniformly accelerated particle (Caianiello *et al.*, 1990).

The extremely large value that $\mathcal{A}_m$ takes for all known particles makes a direct test of the model difficult. Nonetheless a direct test that uses photons in a cavity has also been suggested (Papini *et al.*, 1995). More recently, we have worked out the consequences of the model for the classical electrodynamics of a particle (Feoli *et al.*, 1997), the mass of the Higgs boson (Lambiase *et al.*, 1999; Kuwata, 1996) and the Lamb shift in hydrogenic atoms (Lambiase *et al.*, 1998). In the last instance, the agreement between experimental data and MA corrections is very good for H and D. For He^+ the agreement between theory and experiment is improved by 50% when MA corrections are included. MA effects in muonic atoms appear to be measurable in planned experiments (Chen *et al.*, 1999). MA also affects the helicity and chirality of particles (Chen *et al.*, 2000). Very recently the behaviour of classical (Feoli, *et al.*, 1999a) and quantum (Capozziello *et al.*, 2000a) particles in a Schwarzschild field with MA modifications have been studied.

A limit on the acceleration also occurs in string theory. Here the upper limit manifests itself through Jeans-like instabilities (Sanchez and Veneziano, 1990; Gasperini *et al.*, 1991) which occur when the acceleration induced by the background gravitational field is larger than a critical value $a_c = (m\alpha)^{-1}$for which the string extremities become causally disconnected (Gasperini, 1992). m is the string mass and α is the string tension. Frolov and Sanchez (Frolov and Sanchez, 1991) have then found that a universal critical acceleration a_c must be a general property of strings. It is worth to note that it is possible to derive, in the framework of Caianiello's Quantum Geometry model, the generalized uncertainty principle of string theory (Capozziello *et al.*, 2000b).

The paper is organized as follows. In Section 2 we shortly discuss the Quantum Geometry model and derive the modified Schwarzschild geometry by taking into account the MA corrections (for details see (Feoli *et al.*, 1999a)). In Section 3 we calculate the corrections induced by MA to the quantum mechanical phase of mixed states of neutrinos radially propagating in the modified Schwarzschild geometry. Conclusions are drawn in Sections 4.

2 Modified Schwarzschild Space-Time in Quantum Geometry

The model proposed by Caianiello, which includes the effects of MA in dynamics of particles, was to enlarge the space-time manifold to an eight-dimensional space-time tangent bundle TM_8. In this way the invariant line element is defined as (Caianiello *et al.*, 1990a)

$$d\tilde{s}^2 = g_{AB} dX^A dX^B , \quad A, B = 1, ..., 8 , \tag{2}$$

where the coordinates of TM_8 are

$$X^A = \left(x^\mu ; \frac{1}{\mathcal{A}_m} \frac{dx^\mu}{ds} \right) , \quad \mu = 1, ..., 4 , \tag{3}$$

and

$$g_{AB} = g_{\mu\nu} \otimes g_{\mu\nu} , \quad ds^2 = g_{\mu\nu} dx^\mu dx^\nu . \tag{4}$$

ds is the ordinary line element of the four–dimensional space–time and dx^μ/ds is the four–velocity of the particle moving along its worldline. In Eq. (3), $\mathcal{A}_m$ is the MA depending, in the quantum geometry theory proposed by Caianiello, on the mass m of the particle, whose value is given by $\mathcal{A}_m = 2mc^3/\hbar$. In other models, $\mathcal{A}_m$ is interpreted as an universal constant and m is replaced by the Plank mass m_P. Using Eqs. (3) and (4) the line element (2) can be written as

$$d\tilde{s}^2 = \left(1 + \frac{g_{\mu\nu} \ddot{x}^\mu \ddot{x}^\nu}{\mathcal{A}_m^2} \right) g_{\alpha\beta} dx^\alpha dx^\beta \equiv \sigma^2(x) g_{\alpha\beta} dx^\alpha dx^\beta , \tag{5}$$

where $\ddot{x}^\mu = d^2x^\mu/ds^2$ is the four–acceleration of particles and $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ is the metric due to a background gravitational field. In the absence of gravity, $g_{\mu\nu}$ is replaced by the Minkowski metric tensor $\eta_{\mu\nu}$. The embedding procedure has been developed to find the effective space-time geometry in which a particle can move when the constraint of a MA is present (Caianiello *al.*, 1990b). In fact, if one finds the parametric equations that relate the velocity field $\dot{x}^\mu$ to the first four coordinates x^μ, one can calculate the effective four–dimensional metric on the hypersurface locally embedded in TM_8. This procedure strongly depends on the choice of the velocity field of the particle. From Eq. (5) it follows also that even starting from a phase space TM_8 with a flat metric, i.e. $g_{AB} = \eta_{\mu\nu} \otimes \eta_{\mu\nu}$, in the case of accelerating particles characterized by a velocity field $\dot{x}^\mu$ not trivially constant, one gets an effective four–dimensional geometry which, in general, is curved. In other words, even though the background space–time is flat, the effective geometry experienced by an accelerating particle is curved.

We stress that the curvature of the effective geometry is not induced by matter through the conventional Einstein equations: It is due to the motion in the momentum space and vanishes in the limit $\hbar \to 0$. Thus, it represents a quantum correction to the given background geometry.

In order to calculate the corrections to the Schwarzschild field experienced by a particle initially at infinity and falling toward the origin along a geodesic, one must calculate the metric induced by the embedding procedure (5). On choosing $\theta = \pi/2$, one finds the conformal factor produced by the embedding procedure

$$\sigma^2(r) = 1 + \frac{1}{\mathcal{A}_m^2}\left[\left(1-\frac{2M}{r}\right)\ddot{t}^2 - \frac{\ddot{r}^2}{1-2M/r} - r^2\ddot{\phi}^2\right], \tag{6}$$

where $\ddot{t}, \ddot{r}$ and $\ddot{\phi}$ are given by the standard results (Misner *et al.*, 1973)

$$\begin{aligned}
\ddot{t}^2 &= \frac{\tilde{E}^2}{(1-2M/r)^4}\frac{4M^2}{r^4}\left[\tilde{E}^2 - \left(1-\frac{2M}{r}\right)\left(1+\frac{\tilde{L}^2}{r^2}\right)\right], \\
\ddot{r}^2 &= \left(-\frac{M}{r^2} + \frac{\tilde{L}^2}{r^3} - \frac{3M\tilde{L}^2}{r^4}\right)^2, \\
\ddot{\phi}^2 &= \frac{4\tilde{L}^2}{r^6}\left[\tilde{E}^2 - \left(1-\frac{2M}{r}\right)\left(1+\frac{\tilde{L}^2}{r^2}\right)\right].
\end{aligned} \tag{7}$$

M is the mass of the source, $\tilde{E}$ and $\tilde{L}$ are the total energy (E) and angular momentum (L) per unit of particle mass m. The conformal factor $\sigma^2(r)$ is then given by (Feoli *et al.*, 1999a)

$$\begin{aligned}
\sigma^2(r) = 1 + \frac{1}{\mathcal{A}_m^2}\Bigg\{&-\frac{1}{1-2M/r}\left(-\frac{3M\tilde{L}^2}{r^4} + \frac{\tilde{L}^2}{r^3} - \frac{M}{r^2}\right)^2 + \\
&+\left(-\frac{4\tilde{L}^2}{r^4} + \frac{4\tilde{E}^2M^2}{r^4(1-2M/r)^3}\right)\left[\tilde{E}^2 - \left(1-\frac{2M}{r}\right)\left(1+\frac{\tilde{L}^2}{r^2}\right)\right]\Bigg\}.
\end{aligned} \tag{8}$$

Modifications to the Schwarzschild geometry experienced by radially ($\tilde{L} = 0$) accelerating neutrinos are easily calculated. In fact, from Eq. (8) and by using the weak field approximation, one gets

$$\sigma^2(r) = 1 - \frac{1}{\mathcal{A}_m^2}\left(\frac{1}{4} + \frac{E^2}{m^2} - \frac{E^4}{m^4}\right)\frac{r_s^2}{r^4}, \tag{9}$$

where $r_s = 2GM/c^2$ is Schwarzschild radius.

3 MA Corrections to Quantum Mechanical Phase

Corrections induced by MA to the quantum mechanical phase mixing of massive neutrinos are calculated following Ref. (Bhattacharya *et al.*, 1999). In the semiclassical approximation, i.e. the action of a particle is considered as a quantum phase, a particle propagating in a gravitational field from a point A to a point B, changes its quantum mechanical phase according to the relation (Stodolski, 1979)

$$\Phi = \frac{1}{\hbar}\int_A^B m d\tilde{s} = \frac{1}{\hbar}\int_A^B p_\mu dx^\mu \,. \tag{10}$$

Here $p_\mu = m\tilde{g}_{\mu\nu}(dx^\nu/d\tilde{s})$ is the four–momentum of the particle and $\tilde{g}_{\mu\nu} = \sigma^2(r)g_{\mu\nu}$, where the conformal factor $\sigma^2(x)$ is defined in Eq. (9). In order that different neutrinos could interfere at the same final point B, with coordinates (t_B, r_B), one requires, in the geometrical optical approximation, that the relevant components of the wave function have not started from the same initial point A, with coordinates (t_A, r_A). Then, the quantum mechanical phase becomes

$$\Phi = \frac{1}{\hbar}\int_{r_A}^{r_B} p_r dr \,. \tag{11}$$

Inserting the momentum of the particle, calculated by mass-shell condition $\tilde{g}^{\mu\nu}p_\mu p_\nu = m^2$,

$$p_r = \frac{\sqrt{E^2 - m^2\sigma^2(1 - r_s/r)}}{1 - r_s/r} \tag{12}$$

into Eq. (11) one gets, up to second order in r_s/r,

$$\Phi = \Phi_0 + \Phi_{\mathcal{A}_m} \tag{13}$$

where

$$\Phi_0 = \frac{\sqrt{E^2 - m^2}}{\hbar}(r_B - r_A) + \frac{(2E^2 - m^2)r_s}{2\sqrt{E^2 - m^2}}\log\frac{r_s}{r} + \tag{14}$$
$$-\frac{r_s^2\sqrt{E^2 - m^2}}{\hbar}\left(1 + \frac{m^2}{2(E^2 - m^2)} + \frac{m^4}{8(E^2 - m^2)^2}\right)\left(\frac{1}{r_B} - \frac{1}{r_A}\right) ,$$

represents the result of Ref. (Bhattacharya *et al.*, 1999), and

$$\Phi_{\mathcal{A}_m} = \frac{1}{\mathcal{A}_m^2}\left(\frac{1}{4} + \frac{E^2}{m^2} - \frac{E^4}{m^4}\right)\frac{m^2 r_s^2}{6\hbar\sqrt{E^2 - m^2}}\left(\frac{1}{r_B^3} - \frac{1}{r_A^3}\right) \tag{15}$$

is the contribution due to the MA. For ultra–relativistic neutrinos, $E >> m$, the relative quantum mechanical phase $\Delta\Phi$ of the two different mass eigenstates is given by

$$\Delta\Phi = \Delta\Phi_{(0)} + \Delta\Phi_{\mathcal{A}_m} \,, \tag{16}$$

where

$$\Delta\Phi_{(0)} = \frac{\Delta m^2}{2E\hbar}(r_B - r_A) + \frac{\Delta m^2}{4E^2}(r_B - r_A) - \frac{\Delta m^2(m_1^2 + m_2^2)r_s}{8\hbar E^3}\log\frac{r_B}{r_A} \,, \tag{17}$$

as in Ref. (Bhattacharya *et al.*, 1999), and

$$\Delta\Phi_{\mathcal{A}_m} = \frac{\hbar E^3}{24}\frac{\Delta m^2(m_1^2+m_2^2)}{m_1^4 m_2^4}\frac{r_s^2(r_B^3-r_A^3)}{r_B^3 r_A^3}\,. \tag{18}$$

Here $\Delta m^2 = |m_2^2 - m_1^2|$. In Eq. (17), the first term represents the standard phase of neutrino oscillations, the second term is the kinetic correction to the first order, and finally, the last term is the gravitational correction to the leading order. The second and third term in Eq. (17) can be neglected with respect the first term, so that we will neglect them in what follows. Notice that $\Delta\Phi_{\mathcal{A}_m} \to 0$ as $\hbar \to 0$. It is more convenient to rewrite the phases (17) and (18) in the following way

$$\Delta\Phi_{(0)} = 2.5\cdot 10^3 \frac{\Delta m^2}{\mathrm{eV}^2}\frac{\mathrm{MeV}}{E}\frac{r_A - r_B}{\mathrm{Km}}\,, \tag{19}$$

and

$$\begin{aligned}\Delta\Phi_{\mathcal{A}_m} &= 2.4\cdot 10^8 \frac{\Delta m^2}{\mathrm{eV}^2}\frac{E^3}{\mathrm{MeV}^3}\frac{M^2}{M_\odot^2}\frac{\mathrm{eV}^6}{(m_1 m_2)^4/(m_1^2+m_2^2)}\times \\ &\times \frac{\mathrm{Km}^3}{(r_A r_B)^3/(r_A^3 - r_B^3)}\,, \end{aligned} \tag{20}$$

where $M_\odot$ is the solar mass.

Comparison between the quantum mechanical phases (19) and (20) are reported in Table I for atmospheric neutrinos with mass–squared difference $\Delta m^2 = (10^{-2} \div 10^{-3})\mathrm{eV}^2$. We have assumed the following numerical values: $r_A = R_{Earth} = 6.3\cdot 10^3$Km and $r_B = r_A + 10$Km, $r_s \sim 10^{-6}$Km is the Schwarzschild radius for the Earth and, finally, the energy of neutrinos is $E \sim 1$GeV. MA corrections to the quantum mechanical phase are meaningful for neutrinos with masses $m_1, m_2 \sim 0.05 \div 0.1$eV. In this range, in fact, such corrections turn out to be $10^{-2} \div 10^{-3}$ smaller than the phase (17).

For solar neutrinos, we have a similar situation. Results are summarized in Table II for the values $\Delta m^2 = (10^{-10} \div 10^{-12})\mathrm{eV}^2$, $r_A = R_{Earth}$, $r_B = r_A + 1,5\cdot 10^8$Km, $E \sim 1$MeV and $E \sim 10$MeV, $M \sim M_\odot$. Again, the quantum mechanical phase corrections induced by MA become relevants for neutrino masses of the order $0.05 \div 0.1$eV.

Masses below 0.05eV lead to high corrections that cannot be treated in this perturbative model.

It is worthwhile to point out the different dependence on the energy of the two phases: $\Delta\Phi_{(0)} \sim E^{-1}$ and $\Delta\Phi_{\mathcal{A}_m} \sim E^3$. This can notably help the separation of the two components in experimental tests, because the weight of MA corrections is largely affected by the energy of neutrinos. A good statistical analysis could succeed in bringing this term to light.

4 Conclusions

Einstein's equivalence principle plays a fundamental role in the construction and testing of theories of gravity. Though verified experimentally to better than a part in 10^{11}

for bodies of macroscopic dimensions, doubts have at times been expressed as to its validity down to microscopic scales. It is conceivable, for instance, that the equality of inertial and gravitational mass break down for antimatter, or in quantum field theory at finite temperatures (Donoghue *et al.*, 1984, 1985). Einstein's equivalence principle is also violated in the Quantum Geometry model developed by Caianiello as a first step toward the unification of Quantum Mechanics and General Relativity. The model interprets quantization as curvature of the eight-dimensional space-time tangent bundle TM_8. In this space the standard operators of the Heisenberg algebra are represented as covariant derivatives and the quantum commutation relations are interpreted as components of the curvature tensor.

In this paper we have analyzed the oscillation phenomena of neutrinos propagating in a Schwarzschild geometry modified by the existence of MA, which implies a violation of the equivalence principle. We have calculated the quantum mechanical phase showing that, for the consistence of the Caianiello model, our results are compatible with estimations of the neutrino masses giving $m_\nu \sim 0.05 \div 1$eV.

Eqs. (19) and (20) allow to calculate the flavor oscillation probability, which is given by

$$P_{\nu_e \to \nu_\mu} = \sin^2 2\theta \sin^2 \left(\pi \frac{\Delta r}{\lambda_{\mathcal{A}_m}} \right) , \tag{21}$$

where θ is the mixing angle, and $\lambda_{\mathcal{A}_m}$ is the oscillation length defined as (for simplicity we use the natural units $\hbar = c = 1$)

$$\lambda^{-1}_{\mathcal{A}_m} = \frac{\Delta m^2}{4E\pi} + \frac{E^3}{24\pi} \frac{\Delta m^2 (m_1^2 + m_2^2)}{m_1^4 m_2^4} \frac{r_s^2 (r_A^2 + r_A r_B + r_B^2)}{r_A^3 r_B^3} \, . \tag{22}$$

As well known, in the cases of interest, the oscillation length λ does depend on the energy of neutrinos as $\lambda^{-1} \sim E^n$ (Fogli *et al.*, 1999). Then $\lambda^{-1}_{\mathcal{A}_m}$ corresponds to standard oscillation plus the equivalence principle violation induced by the existence of MA, $(n = -1) \oplus (n = 3)$. The behaviour $\lambda^{-1} \sim E^{-1}$ coming from a flavor depending coupling to gravitational field, as proposed by Gasperini, Leung and Halprin (Gasperini, 1988, 1989; Halprin and Leung, 1991), appeared to fit the SuperKamiokande data, as well as the other alternative mechanisms (Barger *et al.*, 1999; Foot *et al.*, 1998; Chobey and Goswami, 2000). Nevertheless a different analysis of such data, including for example upward-going muons events, has been performed in Refs. (Fogli *et al.*, 1999; Lipari and Lusignolo, 1999). In these papers, it is shown that the best fit does confirm, at least for atmospheric neutrinos, the standard scenario as the dominant oscillation mechanism, whereas the equivalence principle violation, as formulated in (Gasperini, 1988, 1989; Halprin and Leung, 1991), do not provide a viable description of data.

Unlike the mechanism proposed in (Gasperini, 1988, 1989; Halprin and Leung, 1991), in this paper we have suggested an alternative mechanism for introducing, in the framework of Quantum Geometry, a violation of the equivalence principle in the neutrino oscillation physics. The main consequence of this approach, as shown in Eq. (22), is a different behaviour of the inverse of the oscillation length as function of the energy ($\sim E^3$)

with respect to that one obtained in (Gasperini, 1988, 1989; Halprin and Leung, 1991) whose energy dependence has the functional form $(E\Delta f\phi)^{-1}$, where ϕ is the constant gravitational field and Δf the measure of the violation of the equivalence principle.

Acknowledgements

Research supported by MURTS fund PRIN 99.

REFERENCES

Ahluwalia, D.V., and Burgad, C., 1996, *Gen. Rel. Grav.* **28**, 1161; gr-qc/9606031.

Barger, V., Learned, J.G., Pakvasa, S., and Weiler, T.J., 1999, *Phys. Rev. Lett.* **82**, 2640.

Bhattacharya, T., Habib, S., and Mottola, E., 1999, *Phys. Rev.* **D59**, 067301.

Bilenky, S.M. and Pontecorvo, B., 1978, *Phys. Rep.* **41**, 225.

Caianiello, E.R., 1981, *Lett. Nuovo Cimento* **32**, 65.

Caianiello, E.R., De Filippo, S., Marmo, G., and Vilasi, G., 1982, *Lett. Nuovo Cimento* **34**, 112.

Caianiello, E.R., Gasperini, M., and Scarpetta, G., 1990a, *Il Nuovo Cimento* **105B**, 259.

Caianiello, E.R., Feoli, A., Gasperini, M., and Scarpetta, G, 1990b, *Int. J. Theor. Phys.* **29**, 131.

Caianiello, E.R., Gasperini, M, and Scarpetta, G., 1991, *Class. Quant. Grav.* **8** 659.

Caianiello, E.R., 1992, *Rivista del Nuovo Cimento* **15**, No. 4.

Capozziello, S., Lambiase, G., and Scarpetta, G., 1999, *Nuovo Cimento* **114B**, 93.

Capozziello, S., Feoli, A., Lambiase, G., Papini, G., and G. Scarpetta, (2000a) *Phys. Lett.* **A268**, 247.

Capozziello, S., Lambiase, G., Scarpetta, G., (2000b), *Int. J. Theor. Phys.* **39**, 15.

Chen, C.X., Papini, G., Mobed, N., Lambiase, G., and Scarpetta, G., 1999, *Il Nuovo Cimento* **B114**, 199.

Chen, C.X., Papini, G., Mobed, N., Lambiase, G., and Scarpetta, G., 2000, *Il Nuovo Cimento* **B114** 1335.

Choubey, S., and Goswami, S., 2000, *Astropart. Phys.* **14**, 67.

Donoghue, J.F., Holstein, B.R., and Robinett, R.W., 1984, *Phys. Rev.* **D30**, 2561.

Donoghue, J.F., Holstein, B.R., and Robinett, R.W., 1985, *Gen. Rel. Grav.* **17**, 207.

Feoli, A., 1993, *Nucl. Phys.* **B396**, 261.

Feoli, A., Lambiase, G., Papini, G., and Scarpetta, G., 1997. *Il Nuovo Cimento* **112B** 913.

Feoli, A., Lambiase, G., Papini, G. and Scarpetta, G., 1999a, *Phys. Lett.* **A263**, 147.

Feoli, A., Lambiase, G., Nesterenko, V.V., and Scarpetta, G. 1999b, *Phys. Rev.* **D60**, 065001.

Fogli, G.L., Lisi, E., Marrone, A., and Scioscia, G., 1999, *Phys. Rev.* **D60**, 053006.

Foot, R., Leung, C.N., and Yasuda, O., 1998, *Phys. Lett.* **B443**, 185.

Frolov, V.P., Sanchez, N., 1991, *Nucl. Phys.* **B349**, 815.

Gasperini, M., 1988, *Phys. Rev.* **D38**, 2635.

Gasperini, M. 1989, *Phys. Rev.* **D39**, 3606.

Gasperini, M., Sanchez, N., and Veneziano, G., 1991, *Nucl. Phys. B364* 365.
Gasperini, M., 1992, *Gen. Rel. Grav.* **24** 219.
Gibbons, G.W. and Hawking, S.W., 1977, *Phys. Rev.* **D15**, 2738.
Halprin, A. and Leung, C.N., 1991, *Phys. Rev. Lett.* **67**, 1833.
Kuwata, S., 1996, *Il Nuovo Cimento* **B111**, 893.
Lambiase, G., Papini, G., and Scarpetta, G., 1998, *Phys. Lett.* **A244**, 349.
Lambiase, G., Papini, G. and Scarpetta, G., 1999, *Il Nuovo Cimento* **114B**, 189.
Lipari, P. and Lusignoli, M., 1999, *Phys. Rev.* **D60**, 013003.
McGuigan, M., 1994, *Phys. Rev.* **D50**, 5225.
Mikhyev, S.P., and Smirnov, A.Yu, 1986a, *Sov. J. Nucl. Phys.* **42**, 913.
Mikhyev, S.P., and Smirnov, A.Yu, 1986b, *Nuovo Cim.* **C9**, 17.
Misner, C.W., Thorne, K.S., and Wheeler, 1973, J.A.,*Gravitation*, W.H. Freeman and Co. San Francisco.
Papini, G., Feoli, A., and Scarpetta, G., 1995, *Phys. Lett.* **A202** 50.
Sanchez, N., and Veneziano, G., 1990, *Nucl. Phys.* **B333** 253.
Sanchez, N., 1993, *"Structure: from Physics to General Systems"*, Eds. M. Marinaro and G. Scarpetta, World Scientific, Singapore, vol. 1, pag. 118.
Stodolski, L, 1979, *Gen. Rel. Grav.* **11**, 391.
Wolfenstein, L., 1978, *Phys. Rev.* **D17**, 2369.

Table I: Quantum mechanical phase mixing for atmospheric neutrinos with fixed value of Δm^2 and $E \sim 1$GeV. m_1 and m_2 are expressed in eV.

m_1	m_2	Δm^2	$\Delta\Phi_{(0)}$	$\Delta\Phi_{\mathcal{A}_m}$
0.5	0.51	10^{-2}	$2.5 \cdot 10^{-1}$	10^{-8}
0.1	0.14	10^{-2}	$2.5 \cdot 10^{-1}$	$8.6 \cdot 10^{-5}$
0.05	0.11	10^{-2}	$2.5 \cdot 10^{-1}$	$1.8 \cdot 10^{-3}$
0.01	0.1	10^{-2}	$2.5 \cdot 10^{-1}$	1.14
0.5	0.501	10^{-3}	$2.5 \cdot 10^{-2}$	10^{-9}
0.1	0.104	10^{-3}	$2.5 \cdot 10^{-2}$	$2 \cdot 10^{-5}$
0.05	0.06	10^{-3}	$2.5 \cdot 10^{-2}$	$8.9 \cdot 10^{-4}$
0.01	0.03	10^{-3}	$2.5 \cdot 10^{-2}$	1.13

Table II: Quantum mechanical phase mixing for solar neutrinos. Here $m_1 \sim m_2 \sim m$ are expressed in eV and E in MeV.

m	E	$\Delta\Phi_{(0)}/\Delta m^2$	$\Delta\Phi_{\mathcal{A}_m}/\Delta m^2$
0.5	1	$2.5 \cdot 10^{11}$	10^{2}
0.1	1	$2.5 \cdot 10^{11}$	$2 \cdot 10^{6}$
0.05	1	$2.5 \cdot 10^{11}$	$1.2 \cdot 10^{8}$
0.01	1	$2.5 \cdot 10^{11}$	$2 \cdot 10^{12}$
0.5	10	$2.5 \cdot 10^{10}$	10^{5}
0.1	10	$2.5 \cdot 10^{10}$	$2 \cdot 10^{9}$
0.05	10	$2.5 \cdot 10^{10}$	$1.2 \cdot 10^{11}$
0.01	10	$2.5 \cdot 10^{10}$	$2 \cdot 10^{15}$